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M-theory preons cannot arise by quotients

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Abstract: M-theory preons — solutions of eleven-dimensional supergravity preserving 31 supersymmetries — have recently been shown to be locally maximally supersymmetric. This implies that if preons exist they are quotients of maximally supersymmetric solutions. In this paper we show that no such quotients exist. This is achieved by reducing the problem to quotients by cyclic groups in the image of the exponential map, for which there already exists a partial classification, which is completed in the present paper.

Keywords: Space-Time Symmetries, M-Theory, Spacetime Singularities.

Contents

1. Introduction and outline

M-theory preons, solutions of eleven-dimensional supergravity preserving a fraction $\frac{31}{32}$ of the supersymmetry, were conjectured in [1] to be elementary constituents of other BPS states. They have been the subject of much recent work, reviewed for instance in [2], until ultimately serious doubts have been cast over their existence in [3], where it is shown that the gravitino connection of a preonic background is necessarily flat, whence preons are locally maximally supersymmetric. In [4] the earlier analogous result [5] for type IIB supergravity was reinterpreted as a "confinement" of supergravity preons; although a dynamical mechanism which would be responsible for this confinement has not been proposed. In a similar vein one could say that M-theory preons appear to be similarly confined.

The result in [3] implies that the universal cover of a preonic background is maximally supersymmetric or, equivalently, that the putative preonic background must be the quotient of a maximally supersymmetric background by a discrete subgroup of the symmetry group. The aim of this paper is to show that if such a quotient preserves a fraction $\nu \geq \frac{31}{32}$ of the supersymmetry then it is in fact maximally supersymmetric, thus proving conclusively that preons do not exist in eleven-dimensional supergravity or indeed in any known supergravity theory with 32 supercharges.

Let (M, g, F) be a simply-connected maximally supersymmetric background of d=11 supergravity and let G denote the Lie group of F-preserving isometries of the background. We will let $\mathfrak{g}$ denote its Lie algebra. Let $\Gamma < G$ be a discrete subgroup. Then the quotient M/Γ is a (possibly singular) background of eleven-dimensional supergravity which is locally isometric to the original background. A natural question to ask is how much supersymmetry will the quotient preserve. Before we can ask this question, we must ensure that Γ can act on spinors. Since Γ acts by isometries, it preserves the orthonormal frame bundle. The question is whether this action will lift to the spin bundle. We will assume it does. Since (M, g, F) is maximally supersymmetric, the spinor bundle is trivialised by Killing spinors. Let K denote the space of Killing spinors, whence the spinor bundle is isomorphic to $M \times K$, spinor fields are thus smooth maps $M \to K$ and the Killing spinors correspond to the constant maps. Since Γ also preserves F, it preserves the gravitino connection and hence acts on the space of Killing spinors K. The converse is also true: if Γ acts on K then it will act on the spinor bundle by combining the action on K with that on M. If Γ is contained in the identity component of G, then it does act on K and hence on spinors in general. Indeed, the action of the Lie algebra $\mathfrak{g}$ on K is known explicitly, as it is a crucial ingredient in the construction of the superalgebras associated to these backgrounds. Hence we may exponentiate this action to obtain an action of the identity component of G. Should the background possess any "discrete" symmetries; that is, if G has more than one connected component, then we will assume that the group by which we quotient does act on spinors. The supersymmetry preserved by the quotient background is again given by the Killing spinors in the quotient. Since the quotient is locally isometric to M, we may lift this problem to M and we see that Killing spinors of M/Γ correspond to those Killing spinors on M which are Γ-invariant. We will let K^Γ denote the space of Γ-invariant Killing spinors on M. Similarly if $\gamma \in \Gamma$, we will let K^γ denote the space of Killing spinors of M which are left invariant by (the cyclic subgroup generated by) γ.

In this paper we will show that if $\dim K^\Gamma \geq 31$, which we call the 31-*condition*, then $\dim K^\Gamma = 32$, whence it is impossible to construct preonic backgrounds by quotients. The outline of the proof is the following.

We will let $R : G \to \mathrm{GL}(K)$ denote the action of G on K. Suppose that Γ is such that $\dim K^\Gamma = 31$. Then it is plain that for some $\gamma \in \Gamma$, $\dim K^\gamma = 31$. Indeed, were this not the case, then it would mean that either for all $\gamma \in \Gamma$, $\dim K^\gamma = 32$, in which case also $\dim K^\Gamma = 32$; or else for some $\gamma \in \Gamma$, $\dim K^\gamma \leq 30$, whence also $\dim K^\Gamma \leq 30$. It is therefore enough to show that for no $\gamma \in G$, $\dim K^\gamma = 31$. In the next section we will prove this for γ in the image of the exponential map; that is, if $\gamma = \exp(X)$ for some $X \in \mathfrak{g}$, then if $\dim K^\gamma \geq 31$ then $\dim K^\gamma = 32$, whence $R(\gamma) = 1$.

Of course, in most of the groups G under consideration, the exponential map will not be surjective; however we will be able to get around this problem as follows. First let us consider the case where G is connected.

First of all we notice that every element $\gamma \in G$ is the product of a finite number of elements in the image of the exponential map.[1] This is because a finite-dimensional Lie group is multiplicatively generated by any open neighbourhood of the identity and we take one such neighbourhood to be $\exp(S)$ for some open set $0 \in S \subset \mathfrak{g}$. Denoting also by $R : \mathfrak{g} \to \mathrm{End}(K)$ the action of $\mathfrak{g}$ on K, it will follow by inspection of the relevant groups

[1] In fact, it is shown in [6] that every element can be written as the product of at most *two* elements in the image of the exponential map.

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that $\operatorname{tr} R(X) = 0$ for all $X \in \mathfrak{g}$, whence

$$\det R(\exp(X)) = \det e^{R(X)} = e^{\operatorname{tr} R(X)} = 1\,,$$

whence $R : G \to \mathrm{SL}(K)$. Now, if $\dim K^\gamma > 30$, $R(\gamma)$ must lie inside the subgroup of $\mathrm{SL}(K)$ leaving invariant at least 31 linearly independent spinors. In some basis, this is the subgroup

$$\left\{ \begin{pmatrix} I_{31} & \boldsymbol{v} \\ \boldsymbol{0}^t & 1 \end{pmatrix} \middle| \boldsymbol{v} \in \mathbb{R}^{31} \right\}\,,$$

where I_{31} is the 31×31 identity matrix. This means that if for some power k, γ^k lies in the image of the exponential map, then

$$R(\gamma^k) = R(\gamma)^k = \begin{pmatrix} I_{31} & k\boldsymbol{v} \\ \boldsymbol{0}^t & 1 \end{pmatrix} = 1 \iff \boldsymbol{v} = \boldsymbol{0}\,,$$

so that $R(\gamma) = 1$.

The question is thus whether given $\gamma \in G$ some power of γ will lie in the image E_G of the exponential map. This problem turns out to have some history, reviewed, for instance, in [7]. Given $\gamma \in G$, we define its *index (of exponentiality)* by

$$\operatorname{ind}(\gamma) = \begin{cases} \min\left\{k \in \mathbb{N} \middle| \gamma^k \in E_G\right\}\,, & \text{should this exist} \\ \infty & \text{otherwise.} \end{cases}$$

In all maximally supersymmetric backgrounds except for the wave, it is known that every element of G has finite index. Thus the only case which needs to be examined closely is the case where G is the solvable transvection group of a lorentzian symmetric space of Cahen-Wallach [8] type. The (simply-connected) universal covering group $\widetilde{G}$ typically has elements with infinite index. However, $\widetilde{G}$ has no finite-dimensional faithful representations and hence the finite-dimensional representation $R : \widetilde{G} \to \mathrm{SL}(K)$ on Killing spinors will factor through a $\mathbb{Z}$-quotient $\widehat{G}$ of $\widetilde{G}$ for which it will be possible to prove that every element has finite index.

Finally, let us consider the possibility that G is not connected and that Γ is not contained in the identity component G^0 of G. Let $\Gamma^0 = \Gamma \cap G^0$. Then, assuming that G has finitely many components, Γ/Γ^0 is a finite group and the representation $R : \Gamma \to \mathrm{GL}(K)$ factors through $\overline{R} : \Gamma/\Gamma^0 \to \mathrm{GL}(K)$. Since K is a real representation, $\det \overline{R}(\gamma) = \pm 1$. If the determinant is 1, then we can again conclude that $\dim K^\Gamma = 32$.

To prove that $\overline{R} : \Gamma/\Gamma^0 \to \mathrm{SL}(K)$ we can argue as follows. Let $\gamma \in \Gamma \setminus \Gamma^0$. Because the spacetime M is connected, there is some element $\gamma_p \in G^0$ such that $h := \gamma_p^{-1}\gamma$ fixes a point, say, $p \in M$. (Indeed, for every $p \in M$ there will be some $\gamma_p \in G^0$ with this property. Namely, let $q = \gamma \cdot p$ and choose $\gamma_p \in G^0$ such that $q = \gamma_p \cdot p$. Such an element exists because M is connected and hence G^0 already acts transitively.) Now the tangent map $h_* : T_pM \to T_pM$ defines an orthogonal transformation on T_pM which, by hypothesis, lifts to an action on the Killing spinors, and which is induced by restriction from the action of the Pin group. If the spin lift of h_* acts with unit determinant, then from the fact that γ_p

does so as well, it follows that so will γ. It is then a matter of verifying that the Pin group acts with unit determinant on the relevant spinor representation.

The paper is organised as follows. In section 2 we check that no quotient by a cyclic subgroup of symmetries in the image of the exponential map preserves 31 supersymmetries. The result for flat space follows (at least implicitly) from results in [9]. It is discussed in section 2.1 for completeness and because it is the simplest setting in which to present what we call the *even-multiplicity* argument, which is used throughout the paper. The rest of the section is taken by the Freund-Rubin backgrounds, discussed in section 2.2 using the notation of [10], as well as the maximally supersymmetric wave, discussed in section 2.3 for the first time. In section 3 we investigate the surjectivity properties of the exponential map for the symmetry groups of the relevant vacua. A large part of the discussion is devoted to showing that every element of the symmetry group of the maximally supersymmetric wave has finite index. Finally in section 4 we briefly discuss other supergravity theories and conclude that there are no preonic supergravity backgrounds in any known supergravity theory with 32 supercharges.

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2. Cyclic quotients of M-theory vacua

In this section we review the possible quotients of the simply-connected maximally supersymmetric eleven-dimensional supergravity backgrounds by the action of the subgroup generated by an element $\gamma = \exp(X)$ for $X \in \mathfrak{g}$, the Lie algebra of F-preserving isometries of the background. We will show that no quotient preserves exactly a fraction $\frac{31}{32}$ of the supersymmetry.

The method of classification has been explained before in a series of papers [9, 11, 12, 10] to where we refer the reader interested in the details. The basic idea is to study the orbit decomposition of the symmetry Lie algebra $\mathfrak{g}$ under the adjoint action of the Lie group G. Fixing a representative from each orbit, we may then study its action on the Killing spinors. The determination of the adjoint orbits has already been done for all backgrounds but the maximally supersymmetric wave [13, 14], which is the subject of the last subsection. The action on the Killing spinors is a group-theoretical problem which we address in this section.

2.1 Minkowski background

The quotients of $\mathbb{R}^{1,10}$ by continuous cyclic subgroups have been discussed in [9] and the results on discrete quotients follow easily from these. First of all we notice that translations act trivially on spinors, hence the amount of supersymmetry which is preserved by a group element $\gamma \in \mathrm{Spin}(10,1) \ltimes \mathbb{R}^{11}$ is governed by its projection onto $\mathrm{Spin}(10,1)$. Hence from now on we will consider $\gamma \in \mathrm{Spin}(10,1)$ of the form $\gamma = \exp(X)$, for some $X \in \mathfrak{so}(10,1)$. As explained in [9], there are three possible maximal conjugacy classes of such X, depending on the causal type of the vector they leave fixed infinitesimally:

1. $X = \theta_1 \boldsymbol{e}_{12} + \theta_2 \boldsymbol{e}_{34} + \theta_3 \boldsymbol{e}_{56} + \theta_4 \boldsymbol{e}_{78} + \theta_5 \boldsymbol{e}_{9\natural}$,

2. $X = \theta_1 \boldsymbol{e}_{12} + \theta_2 \boldsymbol{e}_{34} + \theta_3 \boldsymbol{e}_{56} + \theta_4 \boldsymbol{e}_{78} + \beta \boldsymbol{e}_{09}$, and

3. $X = \theta_1 \boldsymbol{e}_{12} + \theta_2 \boldsymbol{e}_{34} + \theta_3 \boldsymbol{e}_{56} + \theta_4 \boldsymbol{e}_{78} + \boldsymbol{e}_{+9}$,

where $\boldsymbol{e}_{ij} := \boldsymbol{e}_i \wedge \boldsymbol{e}_j \in \Lambda^2\mathbb{R}^{10,1} \cong \mathfrak{so}(10,1)$, with $\boldsymbol{e}_i$ a pseudo-orthonormal basis for $\mathbb{R}^{10,1}$, and $\boldsymbol{e}_+ := \boldsymbol{e}_0 + \boldsymbol{e}_\natural$, where as usual $\natural$ stands for 10. The Killing spinors of the Minkowski background are isomorphic, as a representation of $\mathrm{Spin}(10,1)$, with the spinor module $\Delta^{10,1}$, which is real and 32-dimensional. We find it convenient to work in the Clifford algebra $\mathrm{C}\ell(10,1)$ which contains the relevant spin group. As an associative algebra, $\mathrm{C}\ell(10,1) \cong \mathrm{Mat}_{32}(\mathbb{C})$, whence it has a unique irreducible module W, which is complex and 32-dimensional. As a representation of $\mathrm{Spin}(10,1)$ it is the complexification of the spinor representation.

The first two cases in the above list are a special case of the following set-up.

2.1.1 The even-multiplicity argument

Let I_a, for $a = 1, \ldots, N$, be commuting real ($I_a^2 = 1$) or complex ($I_a^2 = -1$) structures and consider $R(\gamma) := \exp(\sum_a \frac{1}{2}\theta_a I_a)$ acting on a complex vector space W of dimension 2^N. Since the I_a are commuting, we may diagonalise them simultaneously and decompose

$$W = \bigoplus_{(\sigma_1,\ldots,\sigma_N)\in\mathbb{Z}_2^N} W_{\sigma_1\ldots\sigma_N}$$

where on each one-dimensional $W_{\sigma_1\ldots\sigma_N}$, γ acts by $e^{\sum_a \varepsilon_a\sigma_a\theta_a/2}$, where

$$\varepsilon_a = \begin{cases} 1\,, & \text{if } I_a \text{ is a real structure,} \\ i\,, & \text{if } I_a \text{ is a complex structure.} \end{cases}$$

We now observe that if $R(\gamma)$ acts as the identity on some $W_{\sigma_1\ldots\sigma_N}$, it also acts as the identity on $W_{\bar\sigma_1\ldots\bar\sigma_N}$, where $\bar\sigma_a = -\sigma_a$. This means that the subspace W^γ of γ-invariants has even complex dimension. Being a real representation, it is the complexification of an even-dimensional real subspace. Therefore it cannot be odd-dimensional.

The preceding argument is quite general and will be applied below also in the Freund-Rubin and wave backgrounds. We will refer to it as the *even-multiplicity* argument.

Cases 1 and 2. In these cases, and in the notation of the preceding discussion, $N = 5$ and W is the complexification of $\Delta^{10,1}$, whereas the I_a are the images in the Clifford algebra of the infinitesimal rotations $\boldsymbol{e}_{12}, \boldsymbol{e}_{34}, \boldsymbol{e}_{56}, \boldsymbol{e}_{78}, \boldsymbol{e}_{9\natural}$ or the infinitesimal boost $\boldsymbol{e}_{09}$ in $\mathfrak{so}(10,1)$. Applying the even-multiplicity argument, we see that the γ-invariant subspace is even-dimensional and hence if its dimension is > 30, it must be 32.

Case 3. In this case, the group element is $R(\gamma) = \exp(N + \sum_a \frac{1}{2}\theta_a I_a)$, where N is the image of the infinitesimal null rotation $\boldsymbol{e}_{09} - \boldsymbol{e}_{9\natural} \in \mathfrak{so}(10,1)$ under the spin representation. It follows that $N^2 = 0$ in the Clifford algebra, whence $\exp(N) = 1 + N$ in the spin group. We are after the dimension of the subspace of W consisting of $\psi \in W$ satisfying

$$R(\gamma)\psi = \exp\left(\sum_a \frac{1}{2}\theta_a I_a\right)(1 + N)\psi = \psi\ . \tag{2.1}$$

Let us break up $\psi = \psi_+ + \psi_-$ according to

$$V = V_+ \oplus V_- ,$$

where $V_\pm = \ker(\boldsymbol{e}_0 \pm \boldsymbol{e}_\natural)$, understood as Clifford product. Clearly, $\ker N = \operatorname{Im} N = V_+$. Equation (2.1) becomes

$$\exp\left(\sum_a \frac{1}{2}\theta_a I_a\right)(\psi_+ + \psi_- + N\psi_-) = \psi_+ + \psi_- ,$$

which in turn breaks up into two equations

$$\exp\left(\sum_a \frac{1}{2}\theta_a I_a\right)\psi_- = \psi_- \qquad \text{and} \qquad \exp\left(\sum_a \frac{1}{2}\theta_a I_a\right)(\psi_+ + N\psi_-) = \psi_+ .$$

The even-multiplicity argument says that the invariant space is even dimensional, hence for the 31-condition to hold, the first equation forces $\exp(\sum_a \frac{1}{2}\theta_a I_a) = 1$. The second equation then becomes $N\psi_- = 0$, which means that $\psi_- = 0$. Therefore the most supersymmetry that such a quotient preserves is precisely one half.

2.2 Freund-Rubin backgrounds

The one-parameter quotients of the Freund-Rubin backgrounds, $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$, have been discussed in [10, 15] and the discrete quotients by subgroups in the image of the exponential map have been discussed in [16]. The emphasis in those papers were on quotients which preserve causal regularity. Among such quotients, those by the subgroup generated by (a power of) the generator of the centre of the isometry group of AdS preserve all the supersymmetry. The resulting space is a finite cover of the hyperboloid model for AdS (times the sphere) and admits closed time-like curves. The only other quotients among them preserving more than half of the supersymmetry are described in detail in [17, appendix B] and we will not repeat the calculation here. They preserve fractions $\frac{3}{4}$ and $\frac{9}{16}$ of the supersymmetry. In summary, there are no such quotients preserving exactly $\frac{31}{32}$ of the supersymmetry.

For the present purposes, however, the restriction to causally regular quotients is not desirable and we must therefore revisit the classifications in [10] and study how much supersymmetry is preserved in each case. Every element in the image of the exponential group takes the form $\exp(X)$ for some $X \in \mathfrak{g}$ in the Lie algebra of the symmetry group of the background. Two elements $X, Y \in \mathfrak{g}$ which lie in the same adjoint orbit are equivalent for our purposes since their action on Killing spinors will be related by conjugation and hence, in particular, will leave the same number of Killing spinors invariant. The adjoint orbits have been classified in [10], to whose notation we will adhere in what follows.

2.2.1 $\mathbf{AdS_4} \times S^7$

In this case, the Lie algebra of symmetries is $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)$ and its action on the Killing spinors is given by the tensor product representation $\Delta^{3,2} \otimes \Delta^8_-$, where $\Delta^{3,2}$ is the real 4-dimensional spin representation of $\mathfrak{so}(3,2)$ and Δ^8_- is the real 8-dimensional half-spin

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representation of $\mathfrak{so}(8)$ consisting of negative chirality spinors. The typical element $X \in \mathfrak{g}$ decomposes as $X_A + X_S$, with $X_A \in \mathfrak{so}(3,2)$ and $X_S \in \mathfrak{so}(8)$. Every element $X_S \in \mathfrak{so}(8)$ belongs to some Cartan subalgebra and these are all conjugate. Therefore we may always bring X_S to the form $\theta_1 R_{12} + \theta_2 R_{34} + \theta_3 R_{56} + \theta_4 R_{78}$, where R_{ij} is the element of $\mathfrak{so}(8) = \mathfrak{so}(\mathbb{R}^8)$ which generates rotations in the ij-plane. In contrast there are 15 possible choices for X_A, which are listed in [10, section 4.2.1]. It is natural for the present purposes to treat some of these cases together, which explains the subdivision below.

We will perform our calculations in the Clifford algebra $\mathrm{C}\ell(11,2)$, which contains $G = \mathrm{Spin}(3,2) \times \mathrm{Spin}(8)$, the spin group in question. As an associative algebra, $\mathrm{C}\ell(11,2) \cong \mathrm{Mat}_{64}(\mathbb{C})$ and hence has a unique irreducible module W, which is 64-dimensional and complex and which decomposes under G into the direct sum of 32-dimensional complex subrepresentations (with a real structure) corresponding to the complexifications of $\Delta^{3,2} \otimes \Delta^8_+$ and of $\Delta^{3,2} \otimes \Delta^8_-$. We are interested in the complex subspace $V \subset W$ which is the complexification of $\Delta^{3,2} \otimes \Delta^8_-$. We notice that in many of the cases below we will be able to apply the even-multiplicity argument. There is only one subtlety and that is that we are interested not in W but on a subspace V determined by some chirality condition. In the notation of section 2.1.1, we have $N = 6$ and

$$V = \bigoplus_{\substack{(\sigma_1,\dots,\sigma_6)\in\mathbb{Z}_2^6\\ \sigma_3\sigma_4\sigma_5\sigma_6=-1}} W_{\sigma_1\dots\sigma_6}\,,$$

where the constraints on the signs comes from the chirality condition for $\mathfrak{so}(8)$. To apply the even-multiplicity argument we need to check that if $W_{\sigma_1\dots\sigma_6} \subset V$, then also $W_{\bar\sigma_1\dots\bar\sigma_6} \subset V$, for $\bar\sigma_a = -\sigma_a$. This is clear, though, by definition of the constraints defining V.

In summary, the even-multiplicity argument applies to all cases where X_A consists only of 2×2 blocks in the language of [10]; that is, to case 1, 2, 4, 10, 11, and 12. The remaining cases contain blocks of higher dimension and must be analysed separately. Cases 3,5,14 and 15 are virtually identical to Case 3 in section 2.1 and will not be discussed further.

Cases 6, 7 and 8. These cases are very similar and are defined by the $\mathfrak{so}(3,2)$ component, which can take one of the following forms

- $X_A^{(6)} = -\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34}$,
- $X_A^{(7)} = -\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23})$, and
- $X_A^{(8)} = -\boldsymbol{e}_{12} - \boldsymbol{e}_{13} + \boldsymbol{e}_{24} + \boldsymbol{e}_{34} + \theta(\boldsymbol{e}_{12} + \boldsymbol{e}_{34})$.

We will focus on $X_A^{(8)}$, which will specialise trivially to $X_A^{(6)}$ and leave $X_A^{(7)}$ as a very similar exercise. Let $N + \theta T$ denote the image of $X_A^{(8)}$ in the Clifford algebra, with $N = (\boldsymbol{e}_2 + \boldsymbol{e}_3)(\boldsymbol{e}_1 + \boldsymbol{e}_4)$. It follows that $NT = TN = 0$ and that $N^2 = 0$. Therefore the group element is given by

$$\exp\left(N + \theta T + \sum_{a>2} \theta_a I_a\right) = \exp\left(\sum_a \theta_a I_a\right)(1 + N)\,,$$

where we have put $\theta_1 = \theta_2 = \theta$. We find it convenient to decompose V into four eight-dimensional subspaces

$$V = V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--} ,$$

where $V_{\pm\pm} = \ker(\boldsymbol{e}_2 \pm \boldsymbol{e}_3) \cap \ker(\boldsymbol{e}_1 \pm \boldsymbol{e}_4)$ with uncorrelated signs. Then N acts trivially except on V_{--} where it defines a map $V_{--} \to V_{++}$. Let us write the invariance condition as

$$s \exp(N + T)\psi = \psi ,$$

where $s \in \mathrm{Spin}(8)$. Expanding the exponentials, using that T and N commute and that $N^2 = 0$, we arrive at

$$s \exp(T)(1 + N)\psi = e \exp(T)\psi + s \exp(T)N\psi = \psi \ .$$

The term in the image of N is in V_{++} and because $NT = 0$, $\mathrm{Im}\, T \cap V_{--} = \varnothing$, whence focusing on the V_{--} part of this equation, we find that

$$s\psi_{--} = \psi_{--} \ .$$

The 31-condition forces $s = 1$, and that means that any element of $\mathrm{Spin}(3, 2)$, in particular $\exp(T + N)$, acts with multiplicity 8 on V. Hence the dimension of the invariant subspace is a multiple of 8, which cannot therefore be equal to 31.

Case 9. In this case $X_A = \varphi(\boldsymbol{e}_{12} - \boldsymbol{e}_{34}) + \beta(\boldsymbol{e}_{14} - \boldsymbol{e}_{23}) \in \mathfrak{so}(3, 2)$. The corresponding element $\exp(X_A)$ in $\mathrm{Spin}(3, 2) \subset \mathrm{C}\ell(3, 2)$ is given by $\exp(\frac{1}{2}\varphi A + \frac{1}{2}\beta B)$, where A and B are the images of $\boldsymbol{e}_{12} - \boldsymbol{e}_{34}$ and $\boldsymbol{e}_{14} - \boldsymbol{e}_{23}$, respectively, in the Clifford algebra. It is easy to check that $AB = BA = 0$, whereas $A^2 = -P_+$, $A^3 = -A$, and similarly $B^2 = P_-$ and $B^3 = B$, where $P_\pm = \frac{1}{2}(1 \pm \boldsymbol{e}_{1234})$. Decompose $V = V_+ \oplus V_-$, where $V_\pm = \mathrm{Im}\, P_\pm$. Letting $s = \exp(X_S) \in \mathrm{Spin}(8)$, we want to determine the dimension of the subspace of spinors ψ satisfying

$$s \exp\left(\frac{1}{2}\varphi A\right) \exp\left(\frac{1}{2}\beta B\right)\psi = \psi \ . \tag{2.2}$$

Decomposing $\psi = \psi_+ + \psi_-$, with $\psi_\pm = P_\pm\psi$, we find that the invariance equation (2.2) breaks up into two equations

$$s \exp\left(\frac{1}{2}\varphi A\right)\psi_+ = \psi_+ \qquad \text{and} \qquad s \exp\left(\frac{1}{2}\beta B\right)\psi_- = \psi_- \ .$$

This latter equation implies that $\beta = 0$ for more than half of the supersymmetry to be preserved. The 31-condition then forces $s = 1$, since s acts with even multiplicities. This in turn implies that any element in $\mathrm{Spin}(3, 2)$ acts with multiplicity 8, whence the dimension of the invariant subspace is a multiple of 8 and therefore cannot be equal to 31.

Case 13. Finally, we consider the case where $X_A = \boldsymbol{e}_{12} + \boldsymbol{e}_{13} + \boldsymbol{e}_{15} - \boldsymbol{e}_{24} - \boldsymbol{e}_{34} - \boldsymbol{e}_{45}$. A calculation in the Clifford algebra shows that

$$\exp(X_A) = 1 + (\boldsymbol{e}_1 + \boldsymbol{e}_4)(\boldsymbol{e}_2 + \boldsymbol{e}_3) - \boldsymbol{e}_5(\boldsymbol{e}_1 - \boldsymbol{e}_4) - 2\boldsymbol{e}_{145}(\boldsymbol{e}_2 + \boldsymbol{e}_3) - \tfrac{2}{3}(\boldsymbol{e}_1 - \boldsymbol{e}_4)(\boldsymbol{e}_2 + \boldsymbol{e}_3) \ . \tag{2.3}$$

Letting $V_{\pm\pm} = \ker(\boldsymbol{e}_1 \pm \boldsymbol{e}_4) \cap \ker(\boldsymbol{e}_2 \pm \boldsymbol{e}_3)$ with uncorrelated signs, V decomposes as the direct sum

$$V = V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--} \ .$$

We want to find the dimension of the subspace of ψ satisfying the equation

$$s \exp(X_A)\psi = \psi \,, \tag{2.4}$$

where we have let $s = \exp(X_S) \in \mathrm{Spin}(8)$. Inspection of equation (2.3) reveals that the V_{+-} component of equation (2.4) is simply

$$s\psi_{+-} = \psi_{+-} \,,$$

which, since s acts with even multiplicities, forces $s = 1$ after invoking the 31-condition. This then implies that any element of $\mathrm{Spin}(3,2)$ acts with multiplicity 8 and in particular that the dimension of the subspace of invariants is a multiple of 8 and cannot therefore be equal to 31.

2.2.2 $\mathrm{AdS}_7 \times S^4$

The Lie algebra of symmetries is now $\mathfrak{so}(6,2) \oplus \mathfrak{so}(5)$ and its action on the Killing spinors is given by the underlying real representation of the tensor product representation $\Delta_-^{6,2} \otimes \Delta^5$, where $\Delta_-^{6,2}$ is the quaternionic representation of $\mathfrak{so}(6,2)$ consisting of negative chirality spinors and having complex dimension 8, and Δ^5 is the quaternionic spin representation of $\mathfrak{so}(5)$, which has complex dimension 4. The typical element $X \in \mathfrak{g}$ again decomposes as $X_A + X_S$, with $X_A \in \mathfrak{so}(6,2)$ and $X_S \in \mathfrak{so}(5)$. As before, every element $X_S \in \mathfrak{so}(5)$ belongs to some Cartan subalgebra and may be brought to the form $\theta_1 R_{12} + \theta_2 R_{34}$. In contrast now there are 39 possible choices for X_A, which are listed in [10, section 4.4.1]. It is again natural for the present purposes to treat some of these cases together, which explains the following subdivision.

The general argument at the start of the previous subsection can again be deployed to discard the cases with only 2×2 blocks; that is, rotations or boosts. These are the cases 1, 2, 4, 10, 11, 12, 16, 24, 25, 26, 30, 38 and 39 in [10, section 4.4.1].

Many of the remaining cases already appeared in our discussion of $\mathrm{AdS}_4 \times S^7$ and we will not repeat the arguments here for they are virtually identical to the ones above. These are cases 3, 5, 14, 15, 17, 28, 29 and 31 (which are similar to cases 3, 5, 14 and 15 above); cases 6, 7, 8, 19, 20, 21, 33, 34 and 35 (similar to cases 6,7,8 above); cases 9, 22 and 36 (similar to case 9 above); and cases 13 and 27 (similar to case 13 above). The remaining cases can be subdivided as follows.

Cases 23 and 37. This case involves a double null rotation. The corresponding $X_A = N_1 + N_2 + \theta \boldsymbol{e}_{78}$, where $N_1 = (\boldsymbol{e}_1 + \boldsymbol{e}_4)\boldsymbol{e}_3$ and $N_2 = (\boldsymbol{e}_2 + \boldsymbol{e}_6)\boldsymbol{e}_5$, already written in the Clifford algebra. Notice that $N_1^2 = N_2^2 = 0$ and that $N_1 N_2 = N_2 N_1$, whence $\exp(N_1 + N_2) = (1 + N_1)(1 + N_2)$. We decompose the space V of complexified Killing spinors as

$$V = V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--} \,,$$

where $V_{\pm\pm} = \ker(\boldsymbol{e}_1 \pm \boldsymbol{e}_4) \cap \ker(\boldsymbol{e}_2 \pm \boldsymbol{e}_6)$ with uncorrelated signs. A spinor $\psi \in V$ is invariant if it obeys

$$s(1 + N_1)(1 + N_2)\psi = \psi \,,$$

where $s = R\,(\exp(\theta \boldsymbol{e}_{78}) \exp(X_S))$. The V_{--} component of this equation is

$$s\psi_{--} = \psi_{--} \,,$$

whence, since s acts with even multiplicities, the 31-condition forces $s = 1$. This means that the Spin(5) part of γ acts trivially and so γ acts with multiplicity 4, whence the dimension of the space of invariants is a multiple of 4, which if > 30 must therefore be equal to 32.

Cases 18 and 32. In this case,

$$X_A = \boldsymbol{e}_{15} - \boldsymbol{e}_{35} + \boldsymbol{e}_{26} - \boldsymbol{e}_{46} + \varphi(-\boldsymbol{e}_{12} + \boldsymbol{e}_{34} + \boldsymbol{e}_{56}) + \theta \boldsymbol{e}_{78} \ .$$

We concentrate on the exceptional 6×6 block

$$A + \varphi B = \boldsymbol{e}_{15} - \boldsymbol{e}_{35} + \boldsymbol{e}_{26} - \boldsymbol{e}_{46} + \varphi(-\boldsymbol{e}_{12} + \boldsymbol{e}_{34} + \boldsymbol{e}_{56}) \ .$$

In the Clifford algebra, we find that $AB = BA$, whence in the spin group, $\exp(A + \varphi B) = \exp(A)\exp(\varphi B)$. Therefore we have

$$\exp(X) = \exp(A)\exp(\varphi B + \theta \boldsymbol{e}_{78} + X_S) \ .$$

The second exponential is a semisimple element of the form $\exp(\sum_a \theta_a I_a)$ for commuting complex structures I_a and therefore acts with even multiplicities. Let $W = \Delta^{6,2} \otimes \Delta^5$ be the complex 64-dimensional irreducible module of the Clifford algebra $\mathrm{C}\ell(11, 2)$. Under the action of the six complex structures I_a it decomposes into a direct sum of one-dimensional subspaces

$$W = \bigoplus_{(\sigma_1,\ldots,\sigma_6)\in\mathbb{Z}_2^N} W_{\sigma_1\ldots\sigma_6} \,,$$

whereas the subspace $V = \Delta_-^{6,2} \otimes \Delta^5$ of complexified Killing spinors decomposes as

$$V = \bigoplus_{\substack{(\sigma_1,\ldots,\sigma_6)\in\mathbb{Z}_2^6 \\ \sigma_1\sigma_2\sigma_3\sigma_4=-1}} W_{\sigma_1\ldots\sigma_6} \,,$$

where the constraints on the signs comes from the chirality condition for $\mathfrak{so}(6, 2)$. The semisimple element $s = \exp(\varphi B + \theta \boldsymbol{e}_{78} + X_S)$ preserves each subspace $W_{\sigma_1\ldots\sigma_6} \subset V$ acting on it by the scalar

$$e^{i(\varphi(-\sigma_1+\sigma_2+\sigma_3)+\theta\sigma_4+\theta_5\sigma_5+\theta_6\sigma_6)/2} \ .$$

The spectrum of s on V is therefore given by

- $e^{\pm i(\varphi-\theta+\sigma_5\theta_5+\sigma_6\theta_6)/2}$ each with multiplicity 2—a total of 16;
- $e^{\pm i(\varphi+\theta+\sigma_5\theta_5+\sigma_6\theta_6)/2}$ each with multiplicity 1—a total of 8; and

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- $e^{\pm i(3\varphi+\theta+\sigma_5\theta_5+\sigma_6\theta_6)/2}$ each with multiplicity 1—a total of 8.

Since A commutes with s, $\exp(A)$ preserves each of these eigenspaces. Since $A^3 = 0$, the only possible eigenvalue of e^A is 1. This means that the component of an invariant spinor ψ belonging to any one of the above eigenspaces of s must have eigenvalue 1. The dimension of the eigenspace of s with eigenvalue 1 is even, hence the 31-condition says that this eigenspace must be 32-dimensional, or in other words that $s = 1$, whence the Spin(5) part of γ acts trivially. This implies that γ acts with multiplicity 4, whence the dimension of the space of invariant Killing spinors must be a multiple of 4 and hence, if > 30 it must be 32.

2.3 Maximally supersymmetric wave

The cyclic quotients of the maximally supersymmetric wave have not been worked out before, and we do so here. We will base our discussion of the maximally supersymmetric wave [13] on the paper [14]. In particular, the geometry is that of a lorentzian symmetric space G/H, where the transvection group G and the isotropy subgroup H are described as follows. Let $\mathfrak{g}$ be the 20-dimensional Lie algebra with basis $(\boldsymbol{e}_\pm, \boldsymbol{e}_i, \boldsymbol{e}_i^*)$, for $i = 1, \dots, 9$, and nonzero brackets

$$[\boldsymbol{e}_-, \boldsymbol{e}_i] = \boldsymbol{e}_i^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}_i^*] = -\lambda_i^2 \boldsymbol{e}_i \qquad [\boldsymbol{e}_i^*, \boldsymbol{e}_j] = -\lambda_i^2 \delta_{ij} \boldsymbol{e}_+ ,$$

where

$$\lambda_i = \begin{cases} \frac{\mu}{3}, & i = 1,2,3 \\ \frac{\mu}{6}, & i = 4, \dots, 6 \end{cases} \qquad \text{and } \mu \neq 0. \tag{2.5}$$

Let $\mathfrak{h}$ denote the abelian Lie subalgebra spanned by the $\{\boldsymbol{e}_i^*\}$ and let $H < G$ denote the corresponding Lie subgroup. The obvious subgroup $\mathrm{SO}(3) \times \mathrm{SO}(6) < \mathrm{SO}(9)$ acts as automorphisms on $\mathfrak{g}$ preserving $\mathfrak{h}$ and hence acts as isometries on G/H. Moreover $S := G \rtimes (\mathrm{SO}(3) \times \mathrm{SO}(6))$ preserves the four-form flux, hence it is also the symmetry group of the background.

Let $\mathfrak{s} = \mathfrak{g} \rtimes (\mathfrak{so}(3) \oplus \mathfrak{so}(6))$ denote the Lie algebra of S. Let us examine the possibility of bringing $X \in \mathfrak{s}$ to a normal form via the adjoint action of S. To this end, the following expressions are useful:

$$\begin{aligned} &\mathrm{Ad}\left(e^{t\boldsymbol{e}_-}\right) \boldsymbol{e}_i = \cos(\lambda_i t) \boldsymbol{e}_i + \frac{\sin(\lambda_i t)}{\lambda_i} \boldsymbol{e}_i^* && \mathrm{Ad}\left(e^{t\boldsymbol{e}_i}\right) \boldsymbol{e}_j^* = \boldsymbol{e}_j^* - t\lambda_i^2 \delta_{ij} \boldsymbol{e}_+ \\ &\mathrm{Ad}\left(e^{t\boldsymbol{e}_-}\right) \boldsymbol{e}_i^* = \cos(\lambda_i t) \boldsymbol{e}_i^* + \lambda_i \sin(\lambda_i t) \boldsymbol{e}_i && \mathrm{Ad}\left(e^{t\boldsymbol{e}_i^*}\right) \boldsymbol{e}_- = \boldsymbol{e}_- + \lambda_i^2 t \boldsymbol{e}_i - \frac{1}{2} t^2 \lambda_i^4 \boldsymbol{e}_+ \\ &\mathrm{Ad}\left(e^{t\boldsymbol{e}_i}\right) \boldsymbol{e}_- = \boldsymbol{e}_- - t\boldsymbol{e}_i^* - \frac{1}{2} t^2 \lambda_i^2 \boldsymbol{e}_+ && \mathrm{Ad}\left(e^{t\boldsymbol{e}_i^*}\right) \boldsymbol{e}_j = \boldsymbol{e}_j - t\lambda_i^2 \delta_{ij} \boldsymbol{e}_+ , \end{aligned} \tag{2.6}$$

whereas the adjoint action of $\mathrm{SO}(3) \times \mathrm{SO}(6)$ is the obvious one. Without loss of generality we may assume that the $(\mathfrak{so}(3) \oplus \mathfrak{so}(6))$-component of X lies in the Cartan subalgebra spanned by $\{\boldsymbol{M}_{12}, \boldsymbol{M}_{45}, \boldsymbol{M}_{67}, \boldsymbol{M}_{89}\}$, while still retaining the freedom of acting with the associated maximal torus T, say.

We must now distinguish two cases, depending on whether the component of X along $\boldsymbol{e}_-$ does or does not vanish. If it does not vanish, then from equation (2.6) it follows that we may set the $\boldsymbol{e}_i$-components of X equal to zero by acting with $\mathrm{Ad}\left(e^{t\boldsymbol{e}_i^*}\right)$. Acting with $\mathrm{Ad}\left(e^{t\boldsymbol{e}_3}\right)$ we may shift the $\boldsymbol{e}_3^*$-component to zero. Acting further with T we may rotate in the $(\boldsymbol{e}_1^*, \boldsymbol{e}_2^*)$, $(\boldsymbol{e}_4^*, \boldsymbol{e}_5^*)$, $(\boldsymbol{e}_6^*, \boldsymbol{e}_7^*)$ and $(\boldsymbol{e}_8^*, \boldsymbol{e}_9^*)$ planes to set the $\boldsymbol{e}_i^*$-components to zero for $i = 2, 5, 7, 9$. This brings X to the following form

$$X = v^+\boldsymbol{e}_+ + v^-\boldsymbol{e}_- + v^1\boldsymbol{e}_1^* + v^4\boldsymbol{e}_4^* + v^6\boldsymbol{e}_6^* + v^8\boldsymbol{e}_8^* + \theta^1\boldsymbol{M}_{12} + \theta^2\boldsymbol{M}_{45} + \theta^3\boldsymbol{M}_{67} + \theta^4\boldsymbol{M}_{89}\,, \tag{2.7}$$

where $v^- \neq 0$.

If the $\boldsymbol{e}_-$-component of X vanishes, then we may set the components along some of the $\boldsymbol{e}_i$ to zero, but not much else. This brings X to the form

$$X = v^+\boldsymbol{e}_+ + \sum_i v^i\boldsymbol{e}_i^* + w^1\boldsymbol{e}_1 + w^4\boldsymbol{e}_4 + w^6\boldsymbol{e}_6 + w^8\boldsymbol{e}_8 + \theta^1\boldsymbol{M}_{12} + \theta^2\boldsymbol{M}_{45} + \theta^3\boldsymbol{M}_{67} + \theta^4\boldsymbol{M}_{89}. \tag{2.8}$$

The action of X on the space K of Killing spinors can be read off from the calculation of the superalgebra in [14, section 6]. Let $R : \mathfrak{s} \to \mathrm{End}(K)$ denote the representation, we find

$$R(\boldsymbol{e}_i) = -\frac{1}{2}\lambda_i I\Gamma_i\Gamma_+ \qquad R(\boldsymbol{e}_-) = -\frac{\mu}{4}I\Pi_+ - \frac{\mu}{12}I\Pi_-$$
$$R(\boldsymbol{e}_i^*) = -\frac{1}{2}\lambda_i^2\Gamma_i\Gamma_+ \qquad R(\boldsymbol{M}_{ij}) = \frac{1}{2}\Gamma_{ij}\,,$$

where $\{\Gamma_+, \Gamma_-, \Gamma_i\}$ are the $\mathrm{C}\ell(1,9)$ gamma matrices in a Witt basis, $I = \Gamma_{123}$ and $\Pi_\pm = \frac{1}{2}\Gamma_\pm\Gamma_\mp$ are the projectors onto $\ker\Gamma_\pm$ along $\ker\Gamma_\mp$. We follow the conventions in [14], so that $\Gamma_+\Gamma_- + \Gamma_-\Gamma_+ = 2\mathbf{1}$ and $\Gamma_i^2 = \mathbf{1}$. In particular, whilst $\{R(\boldsymbol{e}_-), R(\boldsymbol{M}_{ij})\}$ are semisimple, $\{R(\boldsymbol{e}_i), R(\boldsymbol{e}_i^*)\}$ are nilpotent. This means that for $X \in \mathfrak{s}$, we may decompose $R(X) = R(X)_S + R(X)_N$ into semisimple and nilpotent parts. Exponentiating and using the BCH formula, we find

$$R(\gamma) := e^{R(X)} = e^{R(X)_S + R(X)_N} = g_S g_N = g_N' g_S\,,$$

where $g_S = e^{R(X)_S}$ and g_N and g_N' are exponentials of nilpotent endomorphisms. In particular, given the nature of the nilpotent endomorphisms in the image of R, we know that $g_N = \mathbf{1} + \alpha\Gamma_+$ and similarly for g_N'.

Before specialising to a particular form of X, let us make some general remarks about the amount of supersymmetry preserved by $R(\gamma)$. The space K of Killing spinors decomposes as $K = K_+ \oplus K_-$, where $K_\pm = K \cap \ker\Gamma_\pm$. Let $\psi = \psi_+ + \psi_-$, with $\psi_\pm \in K_\pm$, be a γ-invariant Killing spinor, so that $R(\gamma)\psi = g_S g_N\psi = \psi$. Decomposing this equation, and using $g_N = \mathbf{1} + \alpha\Gamma_+$, we find

$$g_S g_N(\psi_+ + \psi_-) = g_S(\psi_+ + \psi_- + \alpha\Gamma_+\psi_-) = \psi_+ + \psi_-\ .$$

Since g_S respects the decomposition $K_+ \oplus K_-$, we see that, in particular, $g_S\psi_- = \psi_-$. We would like to estimate how big a subspace of K_- this is.

Let $K^0 \subset K$ denote the subspace of $R(\gamma)$-invariant Killing spinors and let $K_\pm^0 = K^0 \cap K_\pm$. Then letting $K^0 + K_-$ denote the subspace of K generated by K^0 and K_-, we have the fundamental identity

$$\dim(K^0 + K_-) - \dim K^0 = \dim K_- - \dim(K^0 \cap K_-)\ .$$

Since $\dim(K^0 + K_-) - \dim K^0 \leq \operatorname{codim}(K^0 \subset K)$, we arrive at

$$\operatorname{codim}(K^0_- \subset K_-) \leq \operatorname{codim}(K^0 \subset K) \ .$$

If $R(\gamma)$ is to preserve at least $\frac{31}{32}$ of the supersymmetry, then $\operatorname{codim}(K^0 \subset K) \leq 1$, whence $\operatorname{codim}(K^0_- \subset K_-) \leq 1$. Now let $\psi_- \in K^0_-$. We have that $g_S\psi_- = \psi_-$ and $g_N\psi_- = \psi_-$. In particular, the space of g_S-invariants in K_- must have codimension at most 1: it is either 15- or 16-dimensional. We claim that this means that $g_S = 1$. Indeed, g_S is obtained by exponentiating the semisimple part of $R(X)$:

$$\begin{aligned} g_S &= \exp\left(v^- R(\boldsymbol{e}_-) + \theta^1 R(\boldsymbol{M}_{12}) + \theta^2 R(\boldsymbol{M}_{45}) + \theta^3 R(\boldsymbol{M}_{67}) + \theta^4 R(\boldsymbol{M}_{89})\right) \\ &= e^{-\frac{\mu v^-}{4} I\Pi_+} e^{-\frac{\mu v^-}{12} I\Pi_-} e^{\frac{\theta^1}{2}\Gamma_{12}} e^{\frac{\theta^2}{2}\Gamma_{45}} e^{\frac{\theta^3}{2}\Gamma_{67}} e^{\frac{\theta^4}{2}\Gamma_{89}} \ , \end{aligned}$$

whose action on $\psi_- \in K_-$ is given by

$$g_S\psi_- = e^{-\frac{\mu v^-}{12} I} e^{\frac{\theta^1}{2}\Gamma_{12}} e^{\frac{\theta^2}{2}\Gamma_{45}} e^{\frac{\theta^3}{2}\Gamma_{67}} e^{\frac{\theta^4}{2}\Gamma_{89}} \psi_- \ .$$

But now notice that each of the factors in g_S is of the form $e^{\frac{1}{2}\theta^k J_k}$ for commuting complex structures J_k, which can be simultaneously diagonalised upon complexifying K_-. This is precisely the set-up in section 2.1.1, with $W = K_- \otimes_{\mathbb{R}} \mathbb{C}$ and $N = 5$. Therefore we may apply the even-multiplicity argument to conclude that the space of such ψ_- is always divisible by 2, whence g_S cannot preserve exactly 15 such spinors and must in fact preserve all 16.

This means that $g_S = 1$, whence $\theta^i \in 4\pi\mathbb{Z}$ and $\mu v^- \in 24\pi\mathbb{Z}$. The condition on the θ^i say that this part of the group element is trivial, whence the semisimple part of the group element γ is given by

$$\gamma_S = \exp\left(\frac{24\pi k}{\mu}\boldsymbol{e}_-\right) \qquad \text{for some } k \in \mathbb{Z},$$

which acts trivially on the Killing spinors. In addition, it follows from equation (2.6) that this element belongs to the kernel of the adjoint representation and hence to the centre of S.

It remains to show that the nilpotent part of γ cannot preserve precisely a fraction $\frac{31}{32}$ of the supersymmetry. Since γ_S is central, we have that $g_N = \mathbf{1} - \frac{1}{2}\alpha\Gamma_+$, for an endomorphism α given by

$$\alpha = \sum_{i=1}^{9} \left(\lambda_i^2 v^i \Gamma_i + \lambda_i w^i I\Gamma_i\right) \ , \tag{2.9}$$

where the coefficients v^i, w^i are the ones appearing in the expression for $X \in \mathfrak{g}$ in equations (2.7) and (2.8). It is clear from the form of g_N that it acts like the identity on K_+ and that the equation $g_N\psi = \psi$ becomes

$$\left(\mathbf{1} - \frac{1}{2}\alpha\Gamma_+\right)(\psi_+ + \psi_-) = \psi_+ + \psi_- \implies \alpha\Gamma_+\psi_- = 0 \ .$$

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Since Γ_+ has no kernel on K_-, it follows that we must investigate the kernel of α on K_+ or, defining $\breve{\alpha}$ by $\alpha\Gamma_+ = \Gamma_+\breve{\alpha}$, the kernel of

$$\breve{\alpha} = \sum_{i=1}^{9} \left(\lambda_i^2 v^i \Gamma_i - \lambda_i w^i I \Gamma_i\right)$$

on K_-. As discussed above, there are two cases we must consider, corresponding to the forms (2.7) and (2.8) for X.

For X given by equation (2.7), the coefficients $w^i = 0$ and hence $\breve{\alpha} = \sum_{i=1}^{9} \lambda_i^2 v^i \Gamma_i$ is given by the Clifford product by a vector with components $(\lambda_i^2 v_i)$ and by the Clifford relations, provided the vector is nonzero, this endomorphism has trivial kernel. If the vector is zero, so that $v^i = 0$, then we preserve all the supersymmetry. In summary, in this case we may quotient by a subgroup of the centre generated by $\exp(v^+\boldsymbol{e}_+)$, for some v^+, while preserving all of the supersymmetry.

For X given by equation (2.8), we can argue as follows. First of all, since we are already in the situation when $g_S = 1$, we have some more freedom in choosing the normal form. In particular, we may conjugate by $\mathrm{SO}(3) \times \mathrm{SO}(6)$ to set all $w^i = 0$ except for w^1 and w^4. This still leaves an $\mathrm{SO}(2) \times \mathrm{SO}(5)$ which can be used to set $v^{3,6,7,8,9} = 0$. Finally we may conjugate by $e^{t\boldsymbol{e}_-}$ to set $w^4 = 0$, which then gives the further freedom under $\mathrm{SO}(6)$ to set $v^5 = 0$. In summary, and after relabeling, we remain with

$$X = v^+\boldsymbol{e}_+ + v^1\boldsymbol{e}_1^* + v^2\boldsymbol{e}_2^* + v^4\boldsymbol{e}_4^* + w^1\boldsymbol{e}_1 \ .$$

A calculation (performed on computer) shows that the characteristic polynomial of the endomorphism $\breve{\alpha}$ has the form

$$\chi_{\breve{\alpha}}(t) = (t^4 + 2At^2 + B)^4 = \mu_{\breve{\alpha}}(t)^4 \,,$$

where the notation is such that $\mu_{\breve{\alpha}}$ is the minimal polynomial, and A and B are given in terms of $\boldsymbol{z} = (v^1, v^2, v^4, w^1)^t$ and $\boldsymbol{z}^2 = ((v_1)^2, (v^2)^2, (v^4)^2, (w^1)^2)^t$ as

$$A = |\boldsymbol{z}|^2 \qquad \text{and} \qquad B = Q(\boldsymbol{z}^2)\,,$$

where Q is the quadratic form defined by the matrix

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} .$$

This matrix is not positive-definite: it has eigenvalues $0, 2, 1 \pm \sqrt{5}$ with respective eigenvectors

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2}(1 \mp \sqrt{5}) \\ 1 \\ -\frac{1}{2}(1 \mp \sqrt{5}) \\ 1 \end{pmatrix} .$$

Clearly, $\check{\alpha}$ will have nontrivial kernel if and only if $B = 0$, which means $Q(\boldsymbol{z}^2) = 0$. Taking into account that the components of the vector $\boldsymbol{z}^2$ are non-negative, we see that this implies that $v^2 = 0$ and $(v^1)^2 + (v^4)^2 = (w^1)^2$. The characteristic polynomial again satisfies $\chi_{\check{\alpha}}(t) = \mu_{\check{\alpha}}(t)^4$, where now

$$\mu_{\check{\alpha}}(t) = t^2(t^2 + 4(w^1)^2) \ .$$

The dimension of the kernel of $\check{\alpha}$ will be less than the algebraic multiplicity of 0 as a root of the characteristic polynomial. Therefore for $w^1 \neq 0$, the dimension of the kernel will be at most 8 (and, in fact, it is exactly 8). For $w^1 = 0$, the endomorphism $\check{\alpha} \equiv 0$ and the dimension of the kernel is precisely 16.

In summary, we have shown that if an element in the image of the exponential map of the symmetry groups of the maximally supersymmetric vacua preserves at least 31 supersymmetries, it must in fact preserve all 32.

3. Investigating the exponential map

As outlined in the introduction, to complete the proof of the non-existence of preonic quotients, it is crucial to show that every element in the symmetry group of a maximally supersymmetric background has finite index. In other words, that for every element $\gamma \in G$, there is a positive integer n (which may depend on γ) such that γ^n lies in the image of the exponential map. This is a weaker condition than surjectivity of the exponential map. We remind the reader that a Lie group is said to be *exponential* if the exponential map is surjective. It is plain to see that if a simply-connected Lie group is exponential, then so are all connected Lie groups sharing the same Lie algebra. We will find the following partial converse result very useful.

Lemma 1. *Let $\pi : \widehat{G} \to G$ be a finite cover; that is, a surjective homomorphism with finite kernel, with $\widehat{G}$ (and hence G) connected. Then if every element of G has finite index (of exponentiality), so does every element of $\widehat{G}$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of both G and $\widehat{G}$ and let $\exp : \mathfrak{g} \to G$ and $\widehat{\exp} : \mathfrak{g} \to \widehat{G}$ denote the corresponding exponential maps, related by the following commutative diagramme:

$$\begin{tikzcd} & \widehat{G} \arrow[d, "\pi"] \\ \mathfrak{g} \arrow[ur, "\widehat{\exp}"] \arrow[r, "\exp"] & G \end{tikzcd}$$

Since $\pi : \widehat{G} \to G$ is a finite cover, $Z = \ker \pi$ is a finite subgroup of the centre of $\widehat{G}$. (This is because a normal discrete subgroup of a connected Lie group is central.) Now let $\widehat{\gamma} \in \widehat{G}$. Since $\gamma = \pi(\widehat{\gamma})$ has finite index, there exists some positive integer N such that $\gamma^N = \exp(X)$ for some $X \in \mathfrak{g}$. Since

$$\pi(\widehat{\gamma}^N) = \pi(\widehat{\gamma})^N = \gamma^N = \exp(X) = \pi(\widehat{\exp}(X))\,,$$

it follows that there is some $z \in Z$ for which $\widehat{\gamma}^N = z\widehat{\exp}(X)$. Since Z is finite, z has finite order, say, $|z|$, whence

$$\widehat{\gamma}^{N|z|} = \widehat{\exp}(X)^{|z|} = \widehat{\exp}(|z|X)\,,$$

and $\widehat{\gamma}$ also has finite index. $\square$

By virtue of this lemma it is sufficient to exhibit a finite quotient (by a central subgroup) of the groups under consideration for which every element has finite index.

Let us consider the exponential properties of the symmetry groups G of the maximally supersymmetric backgrounds, or more precisely of the related groups which act effectively on the Killing spinors.

The flat backgrounds have symmetry groups $\mathrm{SO}(1, d-1) \ltimes \mathbb{R}^{1,d}$, but as translations act trivially on spinors, it is only $\mathrm{Spin}(1, d-1)$ which concerns us. It follows, for example, from the classification results of [18] that $\mathrm{Spin}(1, 2n)$ for $n \geq 2$ and $\mathrm{Spin}(1, 2m-1)$ for $m \geq 3$ are indeed exponential.

The Freund-Rubin backgrounds have symmetry groups $\widetilde{\mathrm{SO}(2,p)} \times \mathrm{SO}(q)$ for various values of p and q, where $\widetilde{\mathrm{SO}(2,p)}$ is the universal covering group. As discussed, for example, in [10] the groups acting effectively on the Killing spinors are $\mathrm{Spin}(2,p) \times \mathrm{Spin}(q)$. Being compact, $\mathrm{Spin}(q)$ is exponential, whereas $\mathrm{Spin}(2,p)$ is not exponential, but nevertheless, as reviewed in [7], it follows from the results in [19] that the square of every element is in the image of the exponential map.

Finally, the symmetry group of the maximally supersymmetric wave has the form $G \rtimes S$ where G is the solvable transvection group of an 11-dimensional lorentzian symmetric space [8] and $S = \mathrm{SO}(3) \times \mathrm{SO}(6) < \mathrm{SO}(9)$ is a subgroup of the transverse rotation group leaving invariant the fluxes and the matrix A defining the metric. Since S is compact its spin cover, which is the group acting on the Killing spinors, is actually exponential, whence we need only concentrate on the group G.

To understand this case better it pays to look at a toy model. Let $\mathfrak{g}$ denote the four-dimensional Lie algebra with basis $(\boldsymbol{e}_\pm, \boldsymbol{e}, \boldsymbol{e}^*)$ and nonzero brackets

$$[\boldsymbol{e}_-, \boldsymbol{e}] = \boldsymbol{e}^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}^*] = -\boldsymbol{e} \qquad [\boldsymbol{e}, \boldsymbol{e}^*] = \boldsymbol{e}_+ \;.$$

This is the extension of the Heisenberg subalgebra spanned by $(\boldsymbol{e}, \boldsymbol{e}^*, \boldsymbol{e}_+)$ by the outer derivation $\boldsymbol{e}_-$ which acts by infinitesimal rotations in the $(\boldsymbol{e}, \boldsymbol{e}^*)$ plane. It is also known as the Nappi-Witten algebra [20]. It is a Lie subalgebra of $\mathfrak{gl}(4, \mathbb{R})$. Indeed, a possible embedding is given by

$$x\boldsymbol{e}^* + y\boldsymbol{e} + z\boldsymbol{e}_+ + t\boldsymbol{e}_- \mapsto \begin{pmatrix} 0 & t & x & 0 \\ -t & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ -y & x & -2z & 0 \end{pmatrix}, \tag{3.1}$$

and the Lie subgroup $G < \mathrm{GL}(4,\mathbb{R})$ with this Lie algebra is given by

$$G = \left\{ \left. \begin{pmatrix} \cos t & \sin t & x & 0 \\ -\sin t & \cos t & y & 0 \\ 0 & 0 & 1 & 0 \\ -y\cos t - x\sin t & x\cos t - y\sin t & -2z & 1 \end{pmatrix} \right| x,y,z,t \in \mathbb{R} \right\} ,$$

which is diffeomorphic to $S^1 \times \mathbb{R}^3$. Its universal cover $\widetilde{G}$ is diffeomorphic to $\mathbb{R}^4$ and is obtained by unwinding the circle; that is, removing the periodicity of the t-coordinate. Let $g(x,y,z,t)$ denote the group element with coordinates $(x,y,z,t) \in \mathbb{R}^4$. It is convenient, as in [21, section 3.1] but using a different notation, to introduce the complex variable $w = x + iy$ in terms of which the group multiplication on $\widetilde{G}$ is given explicitly by

$$g(w_1, z_1, t_1) g(w_2, z_2, t_2) = g(w', z', t') ,$$

where

$$\begin{aligned} t' &= t_1 + t_2 \\ w' &= w_1 + e^{-it_1} w_2 \\ z' &= z_1 + z_2 - \frac{1}{2} \operatorname{Im}\left(\overline{w}_1 e^{-it_1} w_2 \right) . \end{aligned}$$

The element $g(0,0,t) = \exp(t\boldsymbol{e}_-)$ is central and generates the action of the fundamental group of G on the universal cover $\widetilde{G}$. Any representation of $\widetilde{G}$ for which $\exp(t\boldsymbol{e}_-)$ acts with period $2\pi n$, for some integer n, will factor through an n-fold cover $\widehat{G}$ of G, namely the quotient of $\widetilde{G}$ by the infinite cyclic subgroup of the centre generated by $g(0,0,2\pi n)$. It follows from the Lemma and the fact, to be proven shortly, that G is exponential, that every element of $\widehat{G}$ has finite index.

We now show that that G is exponential. Indeed, the matrix in (3.1) exponentiates inside $\mathrm{GL}(4,\mathbb{R})$ to

$$\begin{pmatrix} \cos t & \sin t & \frac{x\sin t - y(\cos t - 1)}{t} & 0 \\ -\sin t & \cos t & \frac{x(\cos t - 1) + y\sin t}{t} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{-y\sin t + x(\cos t - 1)}{t} & \frac{x\sin t + y(\cos t - 1)}{t} & -2z + \frac{\sin t - t}{t^2}(x^2+y^2) & 1 \end{pmatrix} .$$

It is clear that this is surjective provided that the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{x\sin t - y(\cos t - 1)}{t} \\ \frac{x(\cos t - 1) + y\sin t}{t} \end{pmatrix} = \frac{1}{t} \begin{pmatrix} \sin t & 1 - \cos t \\ \cos t - 1 & \sin t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is nonsingular. Now the above linear transformation is singular precisely when $t \in 2\pi\mathbb{Z}$, but $t \neq 0$. However the group elements

$$\begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ -y & x & -2z & 1 \end{pmatrix}$$

with such a value of t, coincide with those with $t = 0$, for which the above transformation is nonsingular and are hence in the image of the exponential map; explicitly,

$$\begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ -y & x & -2z & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ -y & x & -2z & 0 \end{pmatrix} .$$

Therefore we conclude that the group G is exponential.

The transvection group of a Cahen-Wallach space is a more complicated version of the Nappi-Witten group. The Lie algebra is spanned by $(\boldsymbol{e}_\pm, \boldsymbol{e}_i, \boldsymbol{e}_i^*)$ for $i = 1, \dots, d-2$ with nonzero brackets

$$[\boldsymbol{e}_-, \boldsymbol{e}_i] = \boldsymbol{e}_i^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}_i^*] = A_{ij}\boldsymbol{e}_j \qquad [\boldsymbol{e}_i^*, \boldsymbol{e}_j] = A_{ij}\boldsymbol{e}_+ ,$$

for some non-degenerate symmetric matrix A_{ij}. Although a more general analysis is indeed possible, we shall concentrate uniquely on those matrices A which are negative-definite. Without loss of generality we can rewrite the Lie algebra as

$$[\boldsymbol{e}_-, \boldsymbol{e}_i] = \boldsymbol{e}_i^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}_i^*] = -\lambda_i^2\boldsymbol{e}_i \qquad [\boldsymbol{e}_i^*, \boldsymbol{e}_j] = -\lambda_i^2\delta_{ij}\boldsymbol{e}_+ ,$$

for some $\lambda_i > 0$. It is convenient to change basis $\boldsymbol{e}_i \mapsto \lambda_i^{-1/2}\boldsymbol{e}_i$ and $\boldsymbol{e}_i^* \mapsto \lambda_i^{-3/2}\boldsymbol{e}_i^*$, relative to which the brackets now take a more symmetrical form

$$[\boldsymbol{e}_-, \boldsymbol{e}_i] = \lambda_i\boldsymbol{e}_i^* \qquad [\boldsymbol{e}_-, \boldsymbol{e}_i^*] = -\lambda_i\boldsymbol{e}_i \qquad [\boldsymbol{e}_i^*, \boldsymbol{e}_j] = -\delta_{ij}\boldsymbol{e}_+ .$$

We exhibit this Lie algebra as a subalgebra of $\mathfrak{gl}(2d-2, \mathbb{R})$ via the following embedding

$$\sum_{i=1}^{d-2}(x_i\boldsymbol{e}_i^* + y_i\boldsymbol{e}_i) + t\boldsymbol{e}_- + z\boldsymbol{e}_+ \mapsto \begin{pmatrix} 0 & \lambda_1 t & & & & x_1 & 0 \\ -\lambda_1 t & 0 & & & & y_1 & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & 0 & \lambda_{d-2}t & x_{d-2} & 0 \\ & & & -\lambda_{d-2}t & 0 & y_{d-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -y_1 & x_1 & \cdots & -y_{d-2} & x_{d-2} & -2z & 0 \end{pmatrix} . \tag{3.2}$$

The Lie subgroup $G < \mathrm{GL}(2d-2, \mathbb{R})$ with this Lie algebra consists of matrices of the form

$$\begin{pmatrix} \cos\lambda_1 t & \sin\lambda_1 t & & & & x_1 & 0 \\ -\sin\lambda_1 t & \cos\lambda_1 t & & & & y_1 & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & \cos\lambda_{d-2}t & \sin\lambda_{d-2}t & x_{d-2} & 0 \\ & & & -\sin\lambda_{d-2}t & \cos\lambda_{d-2}t & y_{d-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ u_1 & v_1 & \cdots & u_{d-2} & v_{d-2} & -2z & 1 \end{pmatrix}$$

where

$$u_i = -y_i \cos\lambda_i t - x_i \sin\lambda_i t \qquad v_i = x_i \cos\lambda_i t - y_i \sin\lambda_i t \ .$$

The exponential of the matrix in equation (3.2) is given by

$$\begin{pmatrix} \cos\lambda_1 t & \sin\lambda_1 t & & & & X_1 & 0 \\ -\sin\lambda_1 t & \cos\lambda_1 t & & & & Y_1 & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & \cos\lambda_{d-2} t & \sin\lambda_{d-2} t & X_{d-2} & 0 \\ & & & -\sin\lambda_{d-2} t & \cos\lambda_{d-2} t & Y_{d-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ U_1 & V_1 & \cdots & U_{d-2} & V_{d-2} & -2Z & 1 \end{pmatrix}$$

where

$$\begin{aligned} X_i &= \frac{x_i \sin\lambda_i t + y_i(1-\cos\lambda_i t)}{\lambda_i t} \\ Y_i &= \frac{y_i \sin\lambda_i t - x_i(1-\cos\lambda_i t)}{\lambda_i t} \\ U_i &= -Y_i \cos\lambda_i t - X_i \sin\lambda_i t \\ V_i &= X_i \cos\lambda_i t - Y_i \sin\lambda_i t \\ Z &= z - \frac{1}{2}\sum_{i=1}^{d-2}(x_i^2+y_i^2)\frac{(\lambda_i t - \sin\lambda_i t)}{\lambda_i^2 t^2} \ . \end{aligned} \tag{3.3}$$

We will now specialise further to the geometry of interest, where the λ_i are given by equation (2.5). It is important to observe that the ratios of the λ_i are rational — in fact, integral. This means that whereas the group is not exponential, as we will now see, nevertheless the square of every element lies in the image of the exponential map.

We will now take the λ_i given by equation (2.5). The surjectivity of the exponential map is only in question when the linear map from (x_i, y_i) to (X_i, Y_i) in equation (3.3) fails to be an isomorphism. This happens whenever $\lambda_i t \in 2\pi\mathbb{Z}$ and $t \neq 0$. For the λ_i under consideration, this happens whenever $\mu t \in 6\pi\mathbb{Z}$, but $\mu t \neq 0$. Let $\mu t = 6\pi n$ and $n \neq 0$. Then the group elements with such values of t are given by

$$\begin{pmatrix} 1 & 0 & & & & x_1 & 0 \\ 0 & 1 & & & & y_1 & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & (-1)^n & 0 & x_9 & 0 \\ & & & 0 & (-1)^n & y_9 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ -y_1 & x_1 & \cdots & -(-1)^n y_9 & (-1)^n x_9 & -2z & 1 \end{pmatrix},$$

whence we see that if n is even, this is the same as if $t = 0$ for which the exponential map is surjective, whereas if n is odd, then this is not in the image of the exponential map, but its square is again of the form of the matrices with $t = 0$ and hence in the image of the exponential map. In other words, for every $g \in G$, $g^2 \in E_G$.

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Finally we observed above that the action of $\boldsymbol{e}_-$ on the Killing spinors is such that $\exp(t\boldsymbol{e}_-)$ is periodic, whence the group $\widehat{G}$ acting effectively on spinors is a finite cover of the matrix group G. By the Lemma and the results above, every element in $\widehat{G}$ has finite index.

4. Other supergravity theories

The results in this paper complete the proof of [3] of the non-existence of preonic M-theory backgrounds. How about other supergravity theories?

In [5], it was shown that any solution of IIB supergravity preserving 31 supersymmetries is locally maximally supersymmetric. In the IIB case, this result excludes the possibility of obtaining preons by quotients, because the symmetry groups of the maximally supersymmetric vacua of IIB act complex linearly on Killing spinors, in the conventions where the spinors in IIB are complex chiral spinors. Hence the space of invariants must be a complex subspace and hence must have even dimension.

For IIA supergravity, the result follows from the one for eleven-dimensional supergravity. Indeed, any supergravity background preserving 31 supersymmetries will oxidise to an eleven-dimensional supergravity background preserving at least as much supersymmetry, which by the results of [3] and of the present paper, must in fact be maximally supersymmetric. Furthermore, the IIA background is then a quotient of this eleven-dimensional background by a monoparametric subgroup of symmetries, but as we have shown in this paper, no element in that group preserves exactly 31 supersymmetries, whence if it preserves at least 31 it must preserve them all.

The same argument applies to any other supergravity theory with 32 supercharges obtained by dimensional reduction of $d{=}11$ or IIB supergravities. In summary, there are no supergravity backgrounds, in any known supergravity theory, preserving a fraction $\frac{31}{32}$ of the supersymmetry.

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References

[1] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo and J. Lukierski, *BPS states in M-theory and twistorial constituents*, *Phys. Rev. Lett.* **86** (2001) 4451 [hep-th/0101113].

[2] I.A. Bandos, *BPS preons in supergravity and higher spin theories: an overview from the hill of twistor appraoch*, *AIP Conf. Proc.* **767** (2005) 141 [hep-th/0501115].

[3] U. Gran, J. Gutowski, G. Papadopoulos and D. Roest, $N = 31,\ D = 11$, *JHEP* **02** (2007) 043 [hep-th/0610331].

[4] I.A. Bandos, J.A. de Azcárraga and O. Varela, *On the absence of BPS preonic solutions in IIA and IIB supergravities*, *JHEP* **09** (2006) 009 [hep-th/0607060].

[5] U. Gran, J. Gutowski, G. Papadopoulos and D. Roest, $N = 31$ *is not IIB*, *JHEP* **02** (2007) 044 [hep-th/0606049].

[6] M. Wüstner, *A connected Lie group equals the square of the exponential image*, *J. Lie Theory* **13** (2003) 307.

[7] D. Ž. Doković and K.H. Hofmann, *The surjectivity question for the exponential function of real Lie groups: a status report*, *J. Lie Theory* **7** (1997) 171.

[8] M. Cahen and N. Wallach, *Lorentzian symmetric spaces*, *Bull. Am. Math. Soc.* **76** (1970) 585.

[9] J. Figueroa-O'Farrill and J. Simón, *Generalized supersymmetric fluxbranes*, *JHEP* **12** (2001) 011 [hep-th/0110170].

[10] J. Figueroa-O'Farrill and J. Simón, *Supersymmetric Kaluza-Klein reductions of AdS backgrounds*, *Adv. Theor. Math. Phys.* **8** (2004) 217 [hep-th/0401206].

[11] J. Figueroa-O'Farrill and J. Simón, *Supersymmetric Kaluza-Klein reductions of M2 and M5 branes*, *Adv. Theor. Math. Phys.* **6** (2003) 703 [hep-th/0208107].

[12] J. Figueroa-O'Farrill and J. Simón, *Supersymmetric Kaluza-Klein reductions of M-waves and MKK-monopoles*, *Class. and Quant. Grav.* **19** (2002) 6147 [*Erratum ibid.* **21** (2004) 337] [hep-th/0208108].

[13] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, *Phys. Lett.* **B 134** (1984) 194.

[14] J. Figueroa-O'Farrill and G. Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, *JHEP* **08** (2001) 036 [hep-th/0105308].

[15] O. Madden and S.F. Ross, *Quotients of anti-de Sitter space*, *Phys. Rev.* **D 70** (2004) 026002 [hep-th/0401205].

[16] J. Figueroa-O'Farrill, O. Madden, S.F. Ross and J. Simón, *Quotients of* $\mathrm{AdS}_{p+1} \times S^q$*: causally well-behaved spaces and black holes*, *Phys. Rev.* **D 69** (2004) 124026 [hep-th/0402094].

[17] J. Figueroa-O'Farrill, P. Meessen and S. Philip, *Supersymmetry and homogeneity of M-theory backgrounds*, *Class. and Quant. Grav.* **22** (2005) 207 [hep-th/0409170].

[18] D. Ž. Doković and N.Q. Thǎńg, *On the exponential map of almost simple real algebraic groups*, *J. Lie Theory* **5** (1995) 275.

[19] Y. Sibuya, *Note on real matrices and linear dynamical systems with periodic coefficients*, *J. Math. Anal. Appl.* **1** (1960) 363.

[20] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, *Phys. Rev. Lett.* **71** (1993) 3751 [hep-th/9310112].

[21] J.M. Figueroa-O'Farrill and S. Stanciu, *More D-branes in the nappi-Witten background*, *JHEP* **01** (2000) 024 [hep-th/9909164].